Goodness of fit for Dyadic data

Peter Liu and Shane Lubold

December 11, 2020


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December 2010

Introduction

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$$H_0 : F \in P_0, \quad H_a : F \in P \setminus P_0.$$

i.e. we are interesting in testing if G is drawn from certain graph model P_0.

Methodology

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Theorem 1.

Suppose A is a random symmetric matrix with independent entries such that

$$E(A_{ij}) = 0 \quad \textit{and} \quad Var(\sum_{i \neq j} A_{ij}) = 1.$$

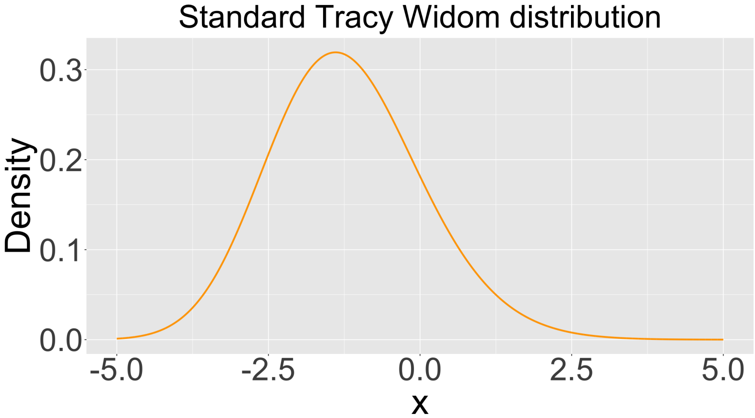
Then

$$n^{2/3}(\lambda_{\max}(A) - 2) \to_d TW_1,$$

$$n^{2/3}(-\lambda_{\min}(A) - 2) \to_d TW_1$$

where TW_1 is the Tracy-Widom distribution with parameter 1.

Tracy Widom distribution



(a) Standard Tracy Widom distribution

Methodology

Theorem 2.

Suppose A is a random asymmetric matrix such that

$$E(A_{ij}) = 0 \quad and \quad Var(A_{ij}) = 1.$$

Then

$$n\sigma_n(A) \rightarrow_d Exp(1).$$

where $\sigma_n(A)$ is the smallest singular value of A.

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- For instance, if we can parameterize a symmetric simple graph G with some probability matrix P such that

$$Pr(G_{ij} = 1|P) = P_{ij},$$

Then the normalized adjacency matrix is given as

$$A_{ij} = \frac{P_{ij} - G_{ij}}{\sqrt{(n-1)P_{ij}(1 - P_{ij})}}.$$

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- The test statistic should converge to Tracy-Widom/exp(1) distribution for valid models.

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- Reject if p-value $< \alpha = 0.05$.

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- But we can fix it via bootstrap!

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Multi-graph	DCSBM	
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- Model that doesn't work: Latent space model.
- We are currently working on ERGM!

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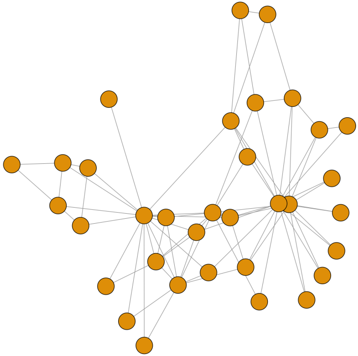
- Arrange the models of interest in a descending order of complexity, i.e. ERGM $\supseteq$ Beta $\supseteq$ SBM $\supseteq$ ER.
- Test starting with the simplest model.
- Our estimate of the model is the first model we fail to reject.
- If all models of interest are rejected, we probably need more complex models to fit our data.

Zachary karate data



(b) Karate club

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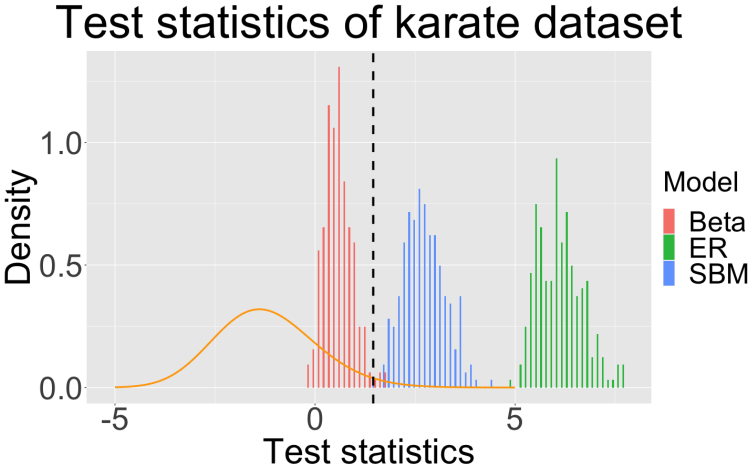

(c) Zachary karate data

Supplementary: Bootstrap correction

1. Given G, compute the MLE $\hat{\boldsymbol{\theta}}$ for a given model P_0.
2. For $b = 1, \ldots, B$:
   2.1 Generate $G_b^\star$ assumed graph model $P_0(\hat{\boldsymbol{\theta}})$.
   2.2 Compute $\lambda_{\max}^\star$ and $\lambda_{\min}^\star$ for $A^\star$:

$$A^\star := \frac{G_b^\star - \hat{P}}{\sqrt{(n-1)\hat{P}(1-\hat{P})}}.$$

3. Reject H_0 when

$$T_{boot} = \mu_{tw} + s_{tw} \max\left(\frac{\lambda_1(\hat{A}) - \hat{\mu}_1}{\hat{s}_1}, -\frac{\lambda_n(\hat{A}) - \hat{\mu}_n}{\hat{s}_n}\right)$$

is bigger than quantile of TW.