Expectations and Sampling Methods

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What we’ve focused on so far...

Monte Carlo Method

Importance Sampling

Rejection Sampling

Bayesian Inference (prior, posterior...)

Maximum Likelihood Estimation

Monte Carlo Method

coin-flipping, card-drawing, needle-tossing…

RV: X, pmf/pdf: fX(x)

$$\mathbb{E}(g(X)) = \sum_{x \in \mathcal{X}} g(x) f_X(x)$$

$$\mathbb{E}(g(X)) = \int_{x \in \mathcal{X}} g(x) f_X(x) dx$$

$$\tilde{g_n}(x) = \frac{1}{n} \sum_{i=1}^{n} g(x_i)$$

MC estimate

$$\lim_{n \to \infty} P\left(|\tilde{g_n}(X) - \mathbb{E}(g(X))| \geq \epsilon\right) = 0.$$

$$\mathbb{E}(\tilde{g_n}(X)) = \mathbb{E}\left(\frac{1}{n} \sum_{i=1}^{n} g(X_i)\right) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}(g(X_i)) = \mathbb{E}(g(X)).$$

unbiased

Importance Sampling

RV: X, density: h(x)

$$\int_{x\in A} g(x)dx = \int_{x\in A} g(x)\frac{h(x)}{h(x)}dx = \int_{x\in A} \frac{g(x)}{h(x)}h(x)dx = \mathbb{E}_h\left(\frac{g(X)}{h(X)}\right)$$

MC Estimator:

$$\widetilde{g_n^h}(X) = \frac{1}{n}\sum_{i=1}^{n}\frac{g(X_i)}{h(X_i)} \quad \text{where} \quad X_i \sim h(x).$$

Rejection Sampling

RV: X, pdf: f(x), "proposal function": g(x)

$$u * M * g(x) \le f(x) \Rightarrow u \le \frac{f(x)}{M * g(x)}$$

Optimal:

$$M = \sup \left\{ \frac{f(x)}{g(x)} \right\}$$

Bayesian Inference

Frequentist Statistics tests whether an event (hypothesis) occurs or not.

P-values, Confidence Intervals

Bayesian Statistics is a mathematical procedure that applies probabilities to statistical problems.

Conditional Probability: $P(A|B) = P(A \cap B)P(B)$

Bayes Theorem: $P(\theta|D)=P(D|\theta)*P(\theta)/P(D)$

$P(\theta)$: prior — $P(D|\theta)$: likelihood

$P(D)$: evidence — $P(\theta|D)$: posterior

Maximum Likelihood Estimation

Assuming a statistical model parameterized by a fixed and unknown θ, the **likelihood function**, L(θ), is the probability of the observed data **x** considered as a function of θ.

$l(\theta) = \ln L(\theta)$

MLE satisfies $dl(\theta)/d\theta=0$

Safety and Efficacy of the BNT162b2 mRNA Covid-19 Vaccine

https://static.us.edusercontent.com/files/607fZpdG97uurk89EkNQaRWX

Vaccine Efficacy against Covid-19 at least 7 days after 2nd dose

Group	Cases	No. subjects
BNT162b2	8	17,411
Placebo	162	17,511
Total	170	34,922

Safety and Efficacy of the BNT162b2 mRNA Covid-19 Vaccine

Beta Binomial Model with Transformation:

$$\theta = \frac{\pi_v}{\pi_v + \pi_p}$$

$\Theta \sim \text{Beta}(0.7000102, 1)$ prior

Claims:

1. Vaccine efficacy $\Psi = (1-2\theta)/(1-\theta) = 95\%$
2. Posterior probability $P(\Psi>30\%) > 0.9999$

Safety and Efficacy of the BNT162b2 mRNA Covid-19 Vaccine

$$X \sim Binom(n_v = 17411, p = \pi_v)$$

$$Y \sim Binom(n_p = 17511, p = \pi_p).$$

W = X|(X+Y=n) :

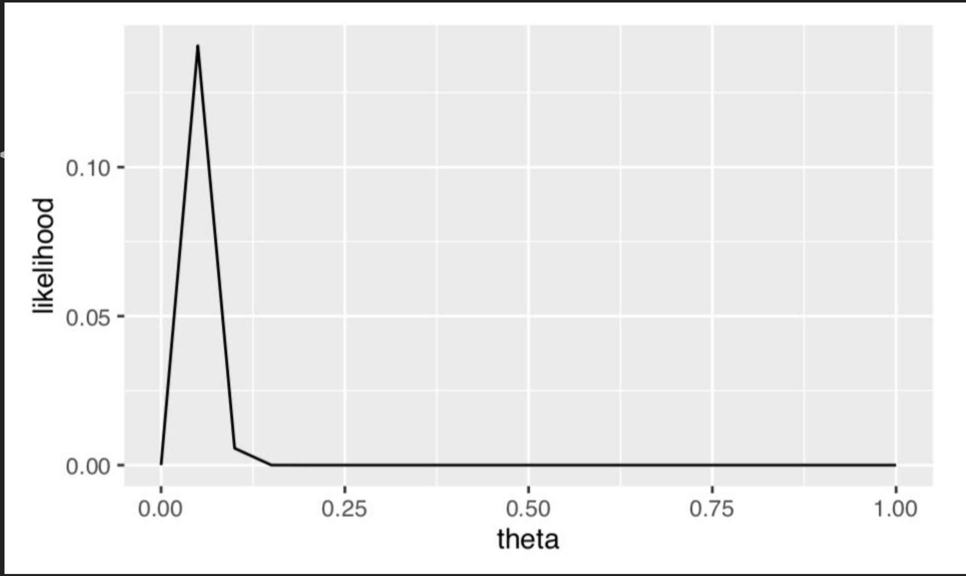
$$W \sim Binom(n = n_v + n_p, \theta = \frac{n_1 \pi_v}{n_v \pi_v + n_p \pi_p})$$

$$W \sim Binom(n = 170, \theta = \frac{\pi_v}{\pi_v + \pi_p})$$

Vaccine Efficacy = 95%

Observation: $w = 8$

$L(\theta)=P(W=8) \propto \theta^8(1-\theta)^{(170-8)}, 0<\theta$



$l(\theta) = 8 \ln \theta + 162 \ln(1 - \theta)$

MLE satisfies $l'(\theta) = 8/\theta - 162/(1-\theta) = 0$

$\hat{\theta} = 4/85 = 0.04706$

$l''(\theta) < 0$

(Newton-Raphson method provides the same result)

Vaccine Efficacy = 95%

H0 : $\theta = 0.0476$ H1 : $\theta \neq 0.0476$

$L(\theta) \propto \theta^{8}(1 - \theta)^{(170-8)}$

$$\lambda = \frac{L(\theta_0)}{L(\hat{\theta})} = \frac{0.0476^{8}(1 - 0.0476)^{162}}{\frac{4}{85}^{8}(1 - \frac{4}{85})^{162}} = 0.999449$$

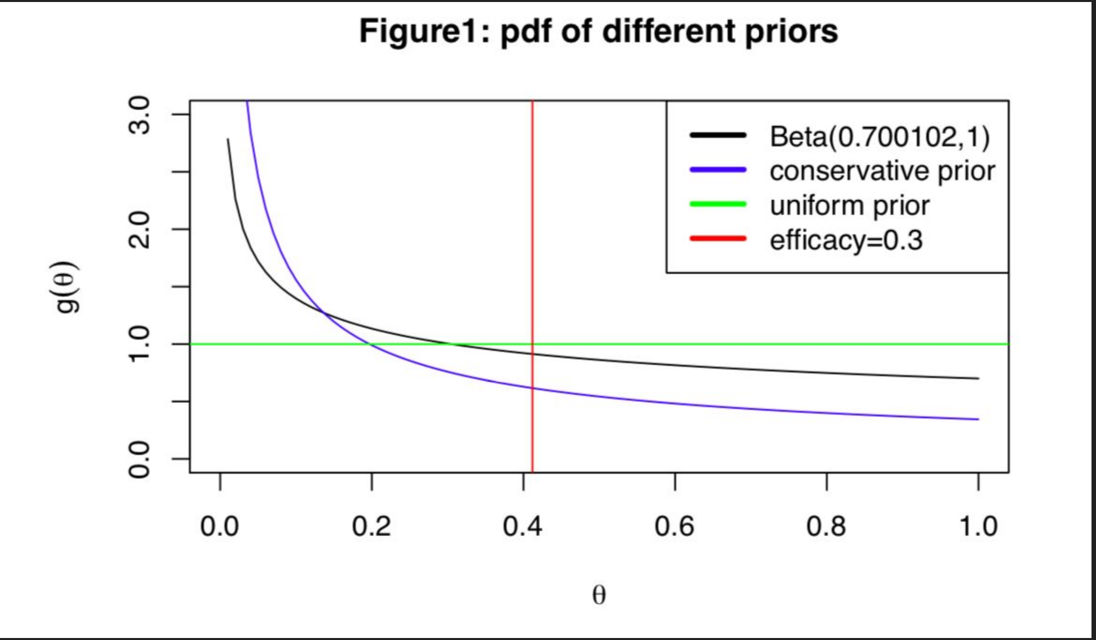
$m = -2 \ln \lambda = 0.001102$

p−value = P(chisq(df = 1) ≥ m) = 1−pchisq(q = m, df = 1) = 0.9735143

Posterior probability P(Ψ>30%) > 0.9999

Corresponding to θ: $P(\theta<=0.4118)>0.9999$

P-value = 5.988e-28<<0.0001



Bayesian Inference

$E(\Psi) = 0.1$

$E(\theta) = (1-0.1)/(2-0.1)$

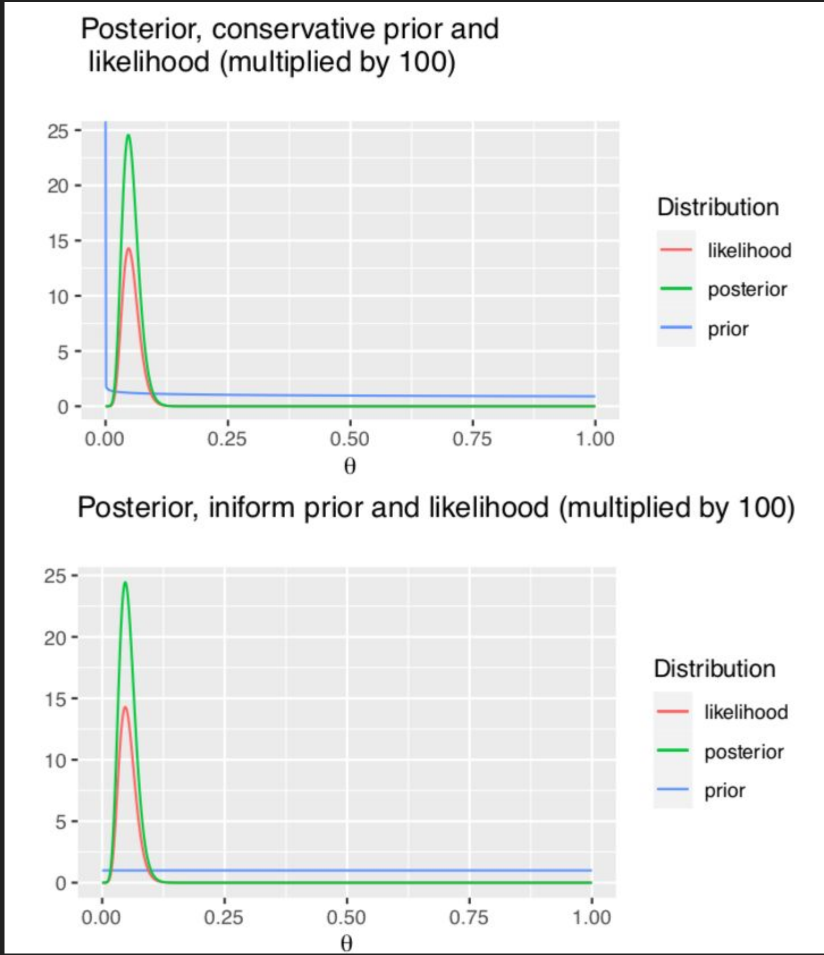
$= \alpha/(\alpha+\beta)$

$\beta = 1$:

new prior Beta(0.9, 1),

uniform prior Beta(1, 1)

Posterior: Beta(0.9+8,1+170-8),
Beta(1+8,1+170-8)



Bayesian Inference

Table 1: Median and 95% HPD interval for θ

Distribution	Median	Lower	Upper	Width
Conservative Prior	0.4629374	0	0.9446	0.9446
Conservative Posterior	0.0501	0.0213	0.0854	0.064
Uniform Prior	0.5	0.05	1	0.95
Uniform Posterior	0.0506	0.0217	0.086	0.0643
Posterior Paper	0.0489	0.0206	0.0839	0.0633

Med = 94.73%

95% posterior interval = [90.67%, 97.82%]

THANK YOU