Random Walks on Graphs

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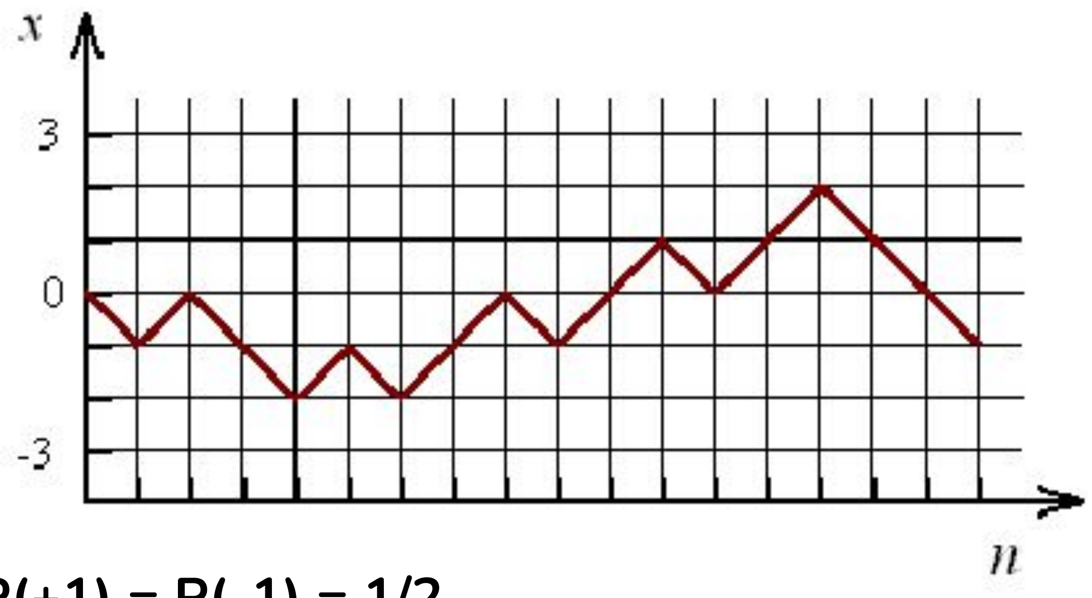
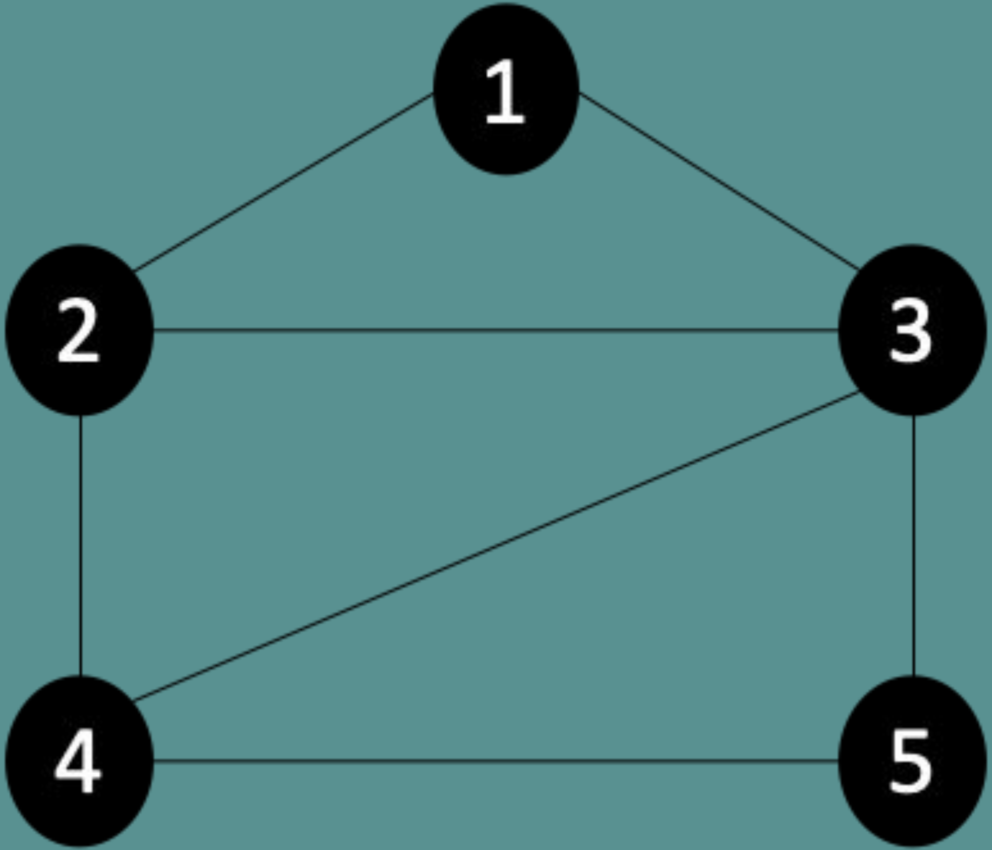
Simple Symmetric Walks



P(+1) = P(-1) = 1/2



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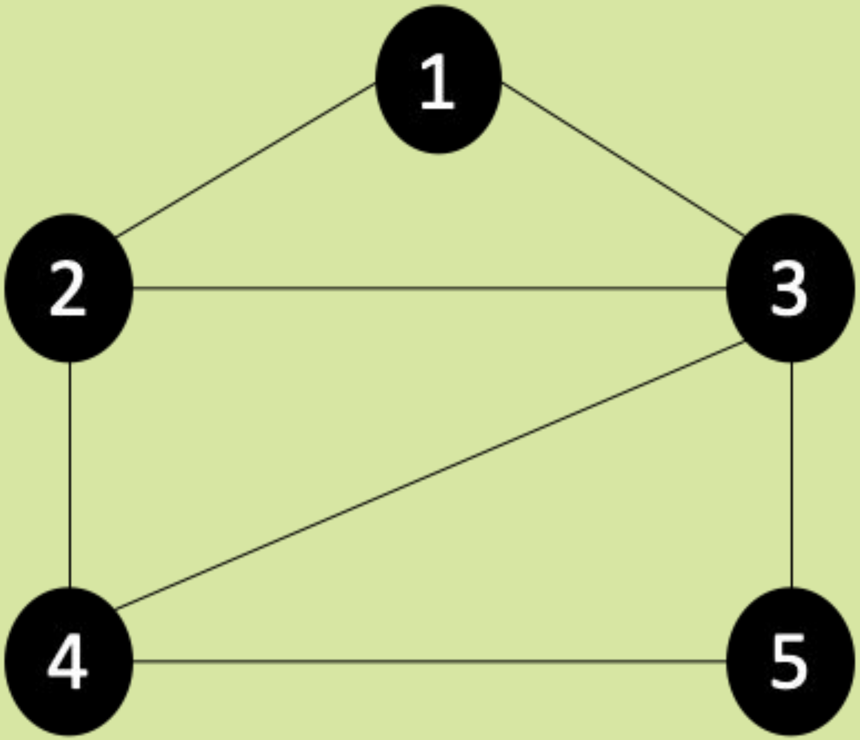


$$M = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

$$M_{ij} = \begin{cases} \frac{1}{deg(v_j)} & if\ (v_i, v_j)\ is\ an\ edge\ in\ the\ graph\ G \\ 0 & otherwise \end{cases}$$

M Matrix After 3 Steps

$$M^3 = \begin{bmatrix} \frac{1}{12} & \frac{41}{144} & \frac{25}{72} & \frac{7}{48} & \frac{5}{36} \\ \frac{41}{216} & \frac{5}{36} & \frac{31}{108} & \frac{31}{108} & \frac{7}{72} \\ \frac{25}{144} & \frac{31}{144} & \frac{2}{9} & \frac{31}{144} & \frac{25}{144} \\ \frac{7}{72} & \frac{31}{108} & \frac{31}{108} & \frac{5}{36} & \frac{41}{216} \\ \frac{5}{36} & \frac{7}{48} & \frac{25}{72} & \frac{41}{144} & \frac{1}{12} \end{bmatrix}$$

Convergence Matrix After 9 Steps

$$M^9 = \begin{bmatrix} 0.1428571 & 0.2142857 & 0.2857143 & 0.2142857 & 0.1428571 \\ 0.1428571 & 0.2142857 & 0.2857143 & 0.2142857 & 0.1428571 \\ 0.1428571 & 0.2142857 & 0.2857143 & 0.2142857 & 0.1428571 \\ 0.1428571 & 0.2142857 & 0.2857143 & 0.2142857 & 0.1428571 \\ 0.1428571 & 0.2142857 & 0.2857143 & 0.2142857 & 0.1428571 \end{bmatrix}$$

Breaking Down the Matrix

$$M = D^{-\frac{1}{2}} S D^{\frac{1}{2}} = D^{-\frac{1}{2}} V \Lambda V^T D^{\frac{1}{2}} = (D^{-\frac{1}{2}} V) \Lambda (D^{\frac{1}{2}} V)^T = \Phi \Lambda \Psi^T$$

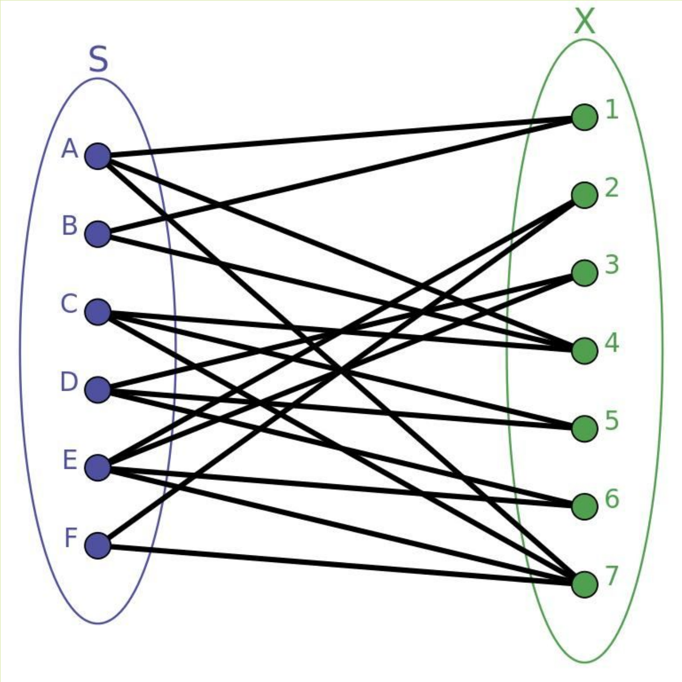
$$M = \sum_{k=1}^{n} \lambda_k \varphi_k \psi_k^T$$

$$M^t = \sum_{k=1}^{n} \lambda_k^t \varphi_k \psi_k^T$$

Issues With Convergence

- Are there matrices that don't converge?
- The implementation of the lazy matrix

$$M' = \frac{1}{2}M + \frac{1}{2}I$$



Laplacian Decomposition of Matrix M

$$L = D - A$$

$$N = D^{-\frac{1}{2}} L D^{-\frac{1}{2}} = I - D^{-\frac{1}{2}} M D^{-\frac{1}{2}}$$

$$W_G = (\frac{1}{2})(I + M_G D_G^{-1})$$

$$W_G = \frac{1}{2} I + \frac{1}{2} M_G D_G^{-1}$$

$$W_G = I - \frac{1}{2} I + \frac{1}{2} M_G D_G^{-1}$$

$$W_G = I - \frac{1}{2} D^{\frac{1}{2}} (I - D_G^{-\frac{1}{2}} M_G D_G^{-\frac{1}{2}}) D_G^{-\frac{1}{2}}$$

$$W_G = I - \frac{1}{2} (D_G^{\frac{1}{2}} I D_G^{-\frac{1}{2}} - D_G^{\frac{1}{2}} D_G^{-\frac{1}{2}} M_G D_G^{-\frac{1}{2}} D_G^{-\frac{1}{2}})$$

$$W_G = I - \frac{1}{2} (I - M_G D_G^{-1})$$

The Stable Distribution

$$\pi = d/(1^T d)$$

$$MD^{-1}\pi = MD^{-1}d/(1^T d) = M1/(1^T d) = d/(1^T d) = \pi$$
$$W\pi = (1/2)I\pi + (1/2)MD^{-1}\pi = (1/2)\pi + (1/2)\pi = \pi$$

$$D^{\frac{1}{2}}c_1\psi_1 = D^{\frac{1}{2}}\frac{1}{(||d^{\frac{1}{2}}||)}\frac{d^{\frac{1}{2}}}{||d^{\frac{1}{2}}||} = \frac{d}{||d^{\frac{1}{2}}||^2} = \frac{d}{\sum_j d(j)} = \pi$$

Dimension Reduction to Show Structure

$$\varphi_t^{(d)}(v_i) = \begin{bmatrix} \lambda_2^t \varphi_2(i) \\ \lambda_3^t \varphi_3(i) \\ \cdots \\ \lambda_{d+1}^t \varphi_{d+1}(i) \end{bmatrix}$$

Resources:

- My Latex Document (https://www.overleaf.com/project/63e42144cf6f571d8332545c)
- First Random Walks Text (https://people.math.osu.edu/husen.1/teaching/571/random_walks.pdf)
- Second Random Walks Text (https://people.math.ethz.ch/~abandeira/BandeiraSingerStrohmer-MDS-draft.pdf)
- Third Random Walks Text (file:///C:/Users/noahm/Downloads/lect10-18_rwG%20(5).pdf)
- My Simulation Code in R (https://github.com/NoahMcMahon1414/STAT_DRP_2023/blob/main/STAT_499_DRP_Simulation.R)