Random Matrix Theory

Wigner’s Semicircle Law

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Introduction

- You're probably familiar with random variables, what about random *matrices*?
- Simply, a random matrix is a matrix whose entries are random variables
- In effect, a random matrix also has random eigenvalues
- Recall, an eigenvalue of a matrix is a scalar factor by which an eigenvector is scaled when the matrix operates on it.

- **What can we say about the distribution of these eigenvalues as the matrix gets larger and larger?**

It's important to restrict the class of random matrices that we study, otherwise the behavior of the eigenvalues could be too unpredictable. Hence, we will only look at Wigner matrices (however arbitrary the properties may seem):

Wigner Matrix (properties):

- Made up of two families of random variables, Y and Z
- Both have zero mean
- the variance of entry $Z_{1,2}$ is equal to 1
- All moments are finite

$$\begin{bmatrix} Y_{11} & Z_{12} & \cdots & Z_{1n} \\ Z_{12} & Y_{22} & \cdots & Z_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ Z_{1n} & Z_{2n} & \cdots & Y_{nn} \end{bmatrix}$$

Let a random Wigner matrix be denoted, as M_n

To properly study the limiting behavior of these eigenvalues, the random matrix must be normalized. Let $X_n = \frac{1}{\sqrt{n}} M_n$ be the *normalized Wigner matrix:*

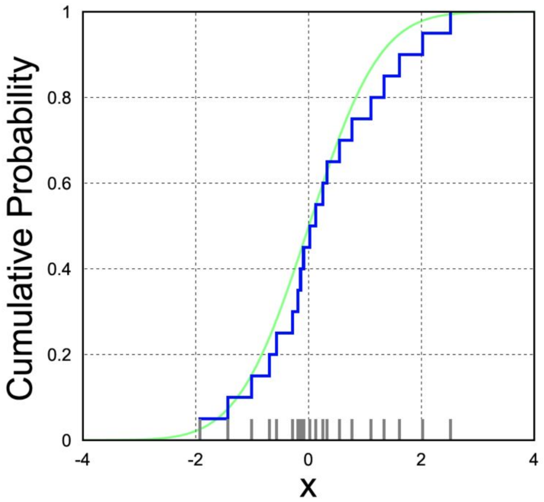
$$X_n = \frac{1}{\sqrt{n}} \begin{bmatrix} Y_{11} & Z_{12} & \cdots & Z_{1n} \\ Z_{12} & Y_{22} & \cdots & Z_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ Z_{1n} & Z_{2n} & \cdots & Y_{nn} \end{bmatrix}$$

- As n grows large, the matrix dimension nxn grows as well
- X_n has *n real eigenvalues*

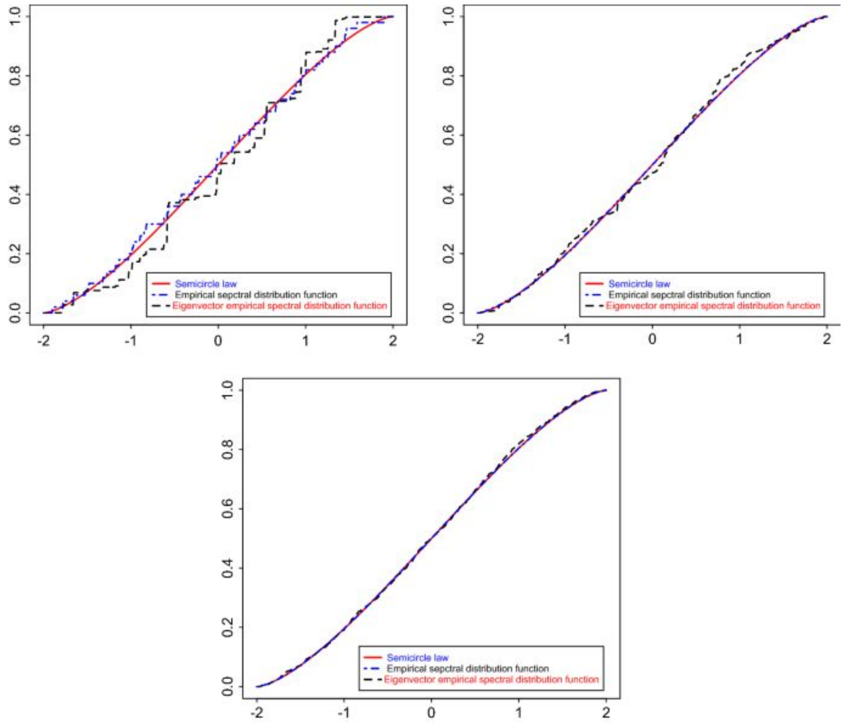
Intuition

- Now, Imagine that we randomly 'draw' an arbitrary random matrix X_n, explicitly determine its n real eigenvalues, and map them into a histogram by their (scaled) cumulative frequency
- This 'histogram' of eigenvalues is called the empirical spectral distribution (ESD):

$$\mu_{X_n}(x) = \frac{1}{n}\sum_{j=1}^{n} 1_{\lambda_j(X_n)\leq x}(x)$$



- Notice, as n grows larger and larger, the ESD (of **observed** eigenvalues) approaches a particular limiting, **theoretical** cdf
- This theoretical cumulative function is called *Wigner's semicircle distribution*

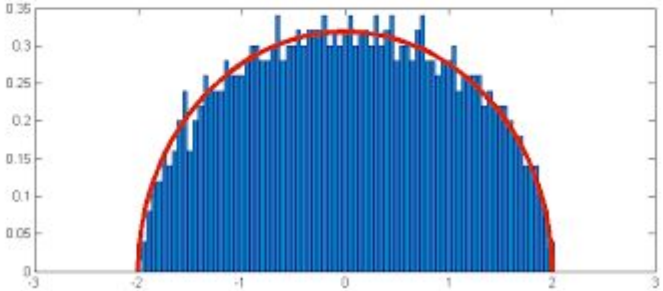


Wigner's Semicircle Distribution:

Theorem 3.1.1. *Let $\{M_n\}_{n=1}^{\infty}$ be a sequence of Wigner matrices, and for each n denote $X_n = M_n/\sqrt{n}$. Then μ_{X_n} converges weakly, in probability to the semicircle distribution,*

$$\sigma(x)dx = \frac{1}{2\pi}\sqrt{4-x^2}\mathbf{1}_{|x|\leq 2}dx. \tag{3.1}$$

where $\sigma(x)dx$ is the **pdf**, or probability density function, which resembles a semi-circle:



Observations

- What do we notice?
- The eigenvalues begin to consistently take on values between [-2, 2] as n grows very large.
- This is why the semicircle distribution has ‘sharp’ edges, which is a surprising result!

Universality of the Semicircle Law

- The semi-circle law is an important theorem of *universality*, which means it works for many different matrix classes (despite our study of convenience using Wigner matrices)
- It's a parallel result to the central limit theorem (CLT) in classical probability, where the sum of random variables approaches a normal distribution regardless of the underlying distribution of the random variables

Proving the semi-circle law using the moments method

- Now that we've hammered in the intuition, the natural question is: how do we rigorously prove that the ESD converges to the semicircle distribution. Or mathematically, $\mu_{x_n} \to \sigma$ as n approaches infinity?

- We'll show convergence of the CDFs, using a proof technique based upon showing convergence in moments (a rare technique for showing convergence)
- In other words, the sample moments of our ESD can be shown to get very close to the actual moments of the semicircle distribution

Final Goal

Proof of Theorem 3.1.1. To conclude $\mu_n \to \sigma$ in the weak sense, we need to show that for any bounded, continuous function $f : \mathbb{R} \to \mathbb{R}$,

$$\mathbb{E}_{\mu_n}[f(X)] \to \mathbb{E}_\sigma[f(X)] \text{ as } n \to \infty$$

- To show the above, we approximate ‘f’ with a polynomial, and then the powers in the polynomials (e.g., X^4) act like moments, which separately we can show converge for all kth moments.
- The details of this proof are quite technical, so we won’t do too deep of a dive during this presentation.
- We’ll still show some of the high-level technical challenges with the proof.

Roadblocks

- Now that we've established in what manner we will prove convergence, we will preemptively get around another roadblock
- Our empirical spectral distribution (ESD) μ_n is the empirical distribution for one *particular* random matrix, and is sensitive to random chance and hard to study.
- We can fix that by introducing the average ESD for all X_n, denoted $\bar{\mu}_n$ and then show that μ_n is not "too different from" $\bar{\mu}_n$ as n grows large.

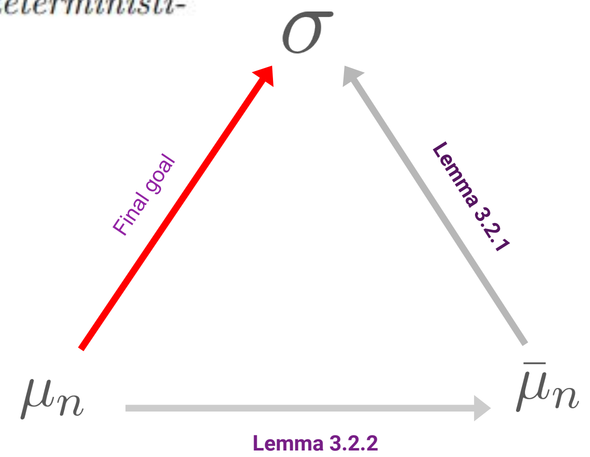
Solution

- We will use all three distributions in our proof

Lemma 3.2.1. *For any positive integer k, $\langle \overline{\mu}_n, x^k \rangle$ converges (deterministically) to $\langle \sigma, x^k \rangle$.*

Lemma 3.2.2. *Fix $\epsilon > 0$ and k a positive integer. Then*

$$\lim_{n\to\infty} P(|\langle \mu_n, x^k \rangle - \langle \overline{\mu}_n, x^k \rangle| > \epsilon) = 0.$$



Thank You!

References

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