**Random Matrix Theory: Wigner's Semicircle Law**

## Introduction

Random matrix theory is the study of random matrices–matrices whose entries are random variables. From linear algebra, one may recall the importance of eigenvalues of a square matrix–or simply put, the collection of scalar factors by which eigenvectors are scaled when that particular matrix operates on it. A random matrix also has these eigenvalues, and an important consequence of the randomness of the matrix is that the eigenvalues are effectively *random* as well. The guiding question is: do these random eigenvalues "follow" some underlying distribution? It turns out that they do.

We begin with the investigation of what happens to the distribution of these eigenvalues when the size of such a matrix becomes arbitrarily large. In other words, what is the asymptotic, or "limiting," behavior of the eigenvalues of a random matrix?

It's important to restrict the class of random matrices that we study, otherwise the behavior of the eigenvalues could be too unpredictable. Hence, we will only look at Wigner matrices. A Wigner matrix is a symmetric matrix which has the following properties:

1) Made up of two families of random variables, Y_i and Z_i,j
2) Both families have zero mean
3) the variance of entry Z_1,2 is equal to 1
4) *all* moments are finite

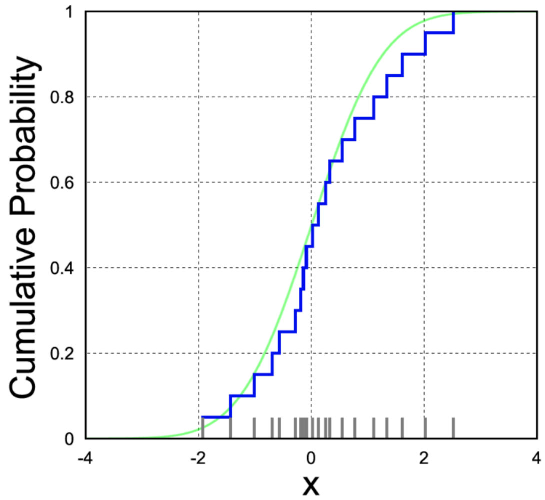
*Note, to properly study the distribution of its observed eigenvalues, we must first normalize this Wigner matrix.

## Intuition

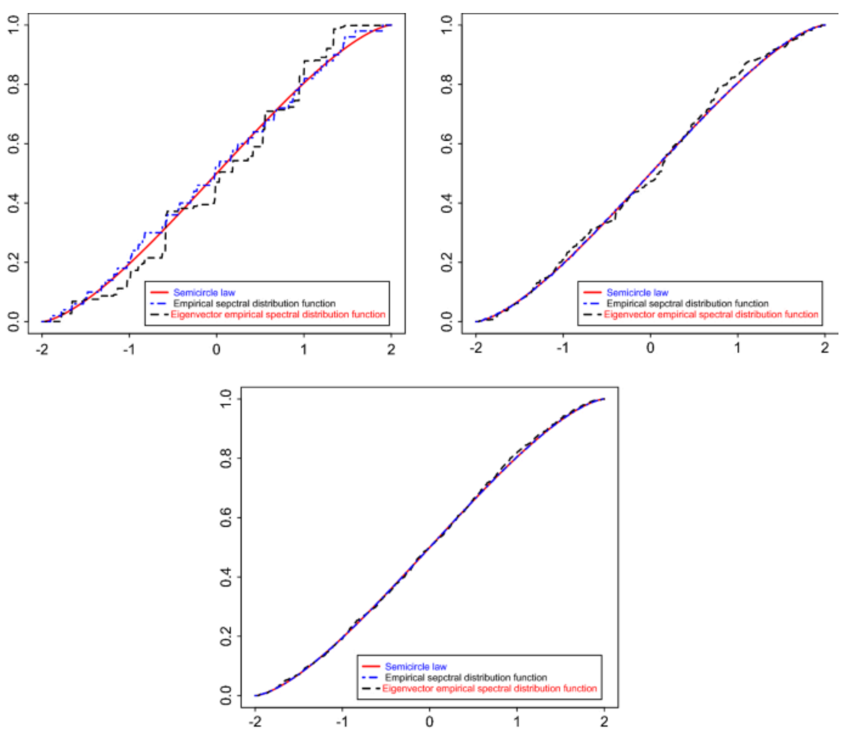
Now, imagine that we randomly "observe" an arbitrary nxn Wigner matrix, explicitly determine its n real eigenvalues (a property of symmetric matrices) and then map them as a histogram by their proportional cumulative frequency. This is represented by the cumulative density function

$$\mu_{X_n}(x) = \frac{1}{n}\sum_{j=1}^{n} 1_{\lambda_j(X_n)\leq x}(x)$$

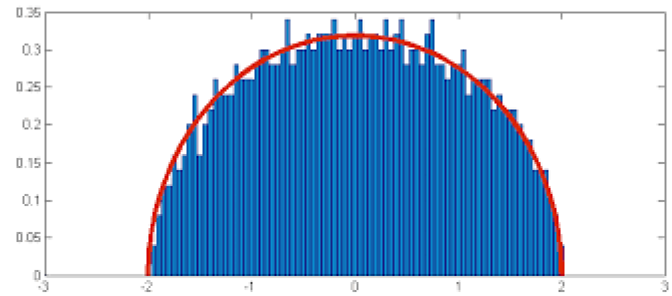
and its respective empirical density plot, which is known as the Empirical Spectral Distribution (ESD). It might look something like below:




As n grows larger and larger, the normalized Wigner matrix also becomes large in dimension. Therefore, the ESD of the n observed eigenvalues will begin to converge toward a limiting, *theoretical* distribution, known as Wigner's semicircle law. This gradual process is illustrated by the following three density plots, with each successive plot representing an increasingly large value of n. Notice how the dotted *blue* line gradually begins to resemble the unchanging red line:



By differentiating the cumulative density function, we can obtain a probability density function whose distribution resembles a perfect semicircle with diameter extending between -2 and 2.




**Universality**

One may recall the famous and ubiquitous result known as the Central Limit Theorem (CLT) in classical probability theory and statistics. It roughly states that the sum of n random variables approaches a normal distribution *regardless* of the underlying distribution of the random variables themselves. This property is known as universality.

The semicircle law is analogous to the CLT within the realm of random matrix theory and free probability, and is also an important theorem of universality. Though, in this case, universality suggests that the semi-circle law applies to many different matrix classes (despite our study of convenience using symmetric Wigner matrices).

**Proving the semi-circle law using the moments method**

Now that we've hammered in the intuition, the natural question is: how do we rigorously prove that the ESD converges to the semicircle distribution? A high-level overview of the proof will be outlined below.

The main strategy is to show convergence of the cumulative density functions using a proof technique based upon showing convergence in moments. In other words, the sample moments (of the ESD) can be shown to get very close to the actual moments (of the semicircle distribution). The formalized statement is as follows:

*Proof of Theorem 3.1.1.* To conclude $\mu_n \to \sigma$ in the weak sense, we need to show that for any bounded, continuous function $f : \mathbb{R} \to \mathbb{R}$,

$$\mathbb{E}_{\mu_n}[f(X)] \to \mathbb{E}_{\sigma}[f(X)] \text{ as } n \to \infty$$

To show the above, we approximate ‘f’ with a polynomial, and then the powers in the polynomials (e.g., X^4) act like moments, which separately we can show converge for all kth moments.

Because the details of this proof are quite technical, the actual content of the proof won’t be shown. If you would like to read more into the proof, check out *Methods of Proof in Random Matrix Theory* by Adina Roxana Feier.

**References**

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