Measure and Concentration

Bhaumik and Jian

Introduction

In 1829, Dirichlet proved the following theorem:

Theorem (Dirichlet, 1829)

The Fourier series of a piecewise-smooth f on $[-\pi, \pi]$ converges to

$$\frac{f(x_+) + f(x_-)}{2}.$$

That is, the Fourier series of f converges at every point of continuity. At discontinuities, it takes the middle value.

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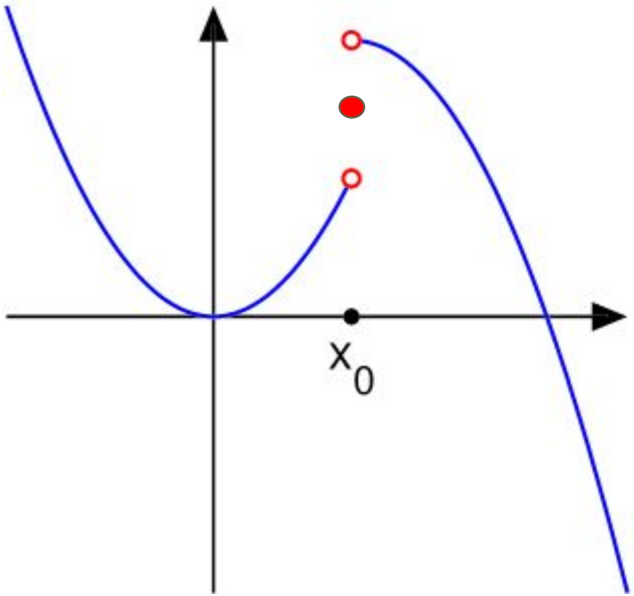

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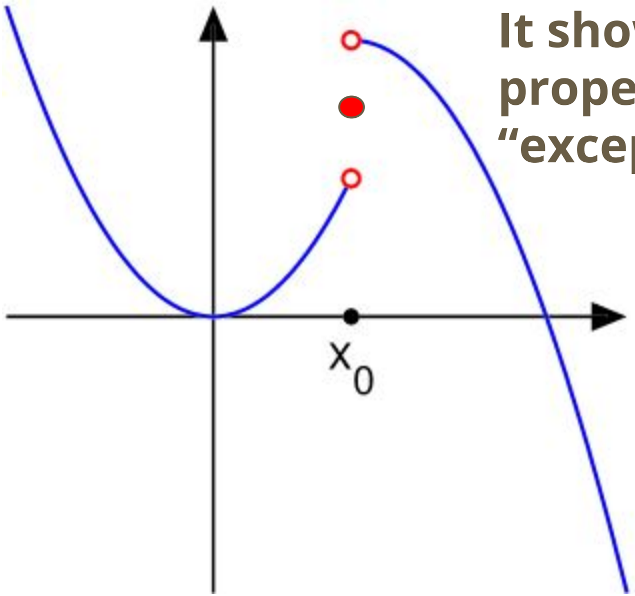

It showed that some desired properties do not hold at "exceptional yet negligible sets."

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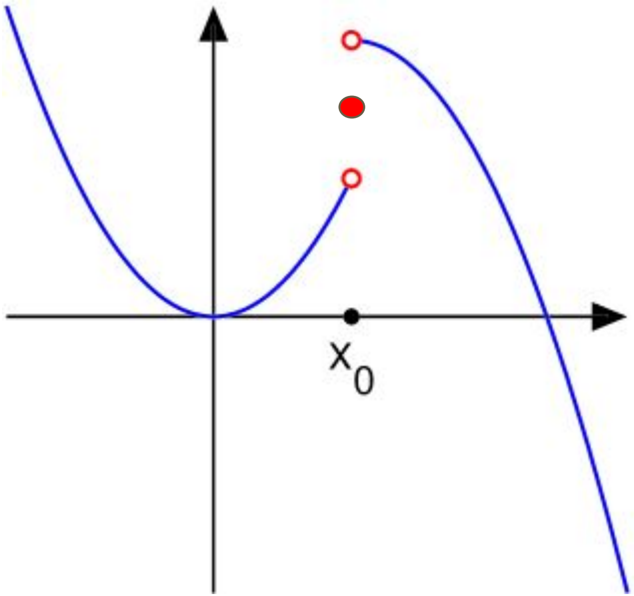
This made it clear to analysts that to rigorously establish convergence results in general, we need to be able to rule out “exceptional yet negligible sets.”



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We simplify to the case of coin flips:

$$\frac{number\ of\ heads}{number\ of\ flips} \longrightarrow \frac{1}{2}$$

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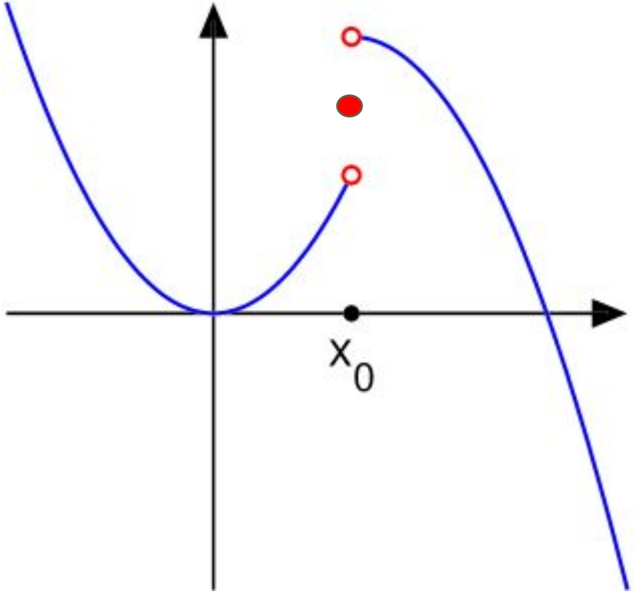
What happens when we only flip heads????

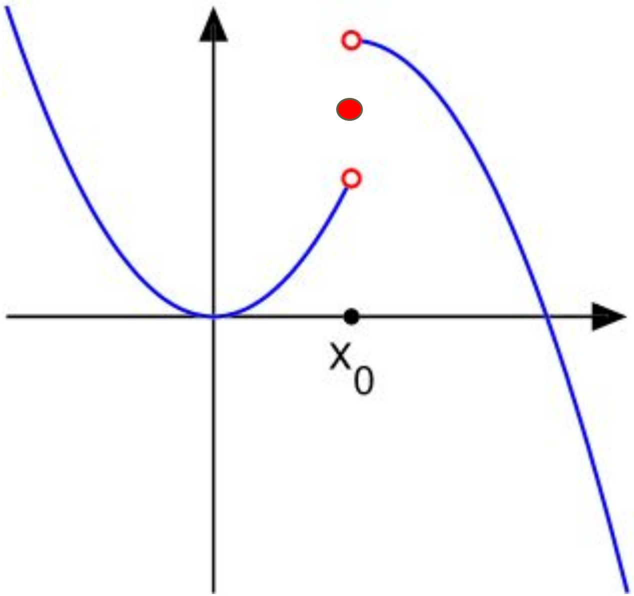
Why measure theory in probability?

Recall the idea of “exceptional yet negligible sets.”



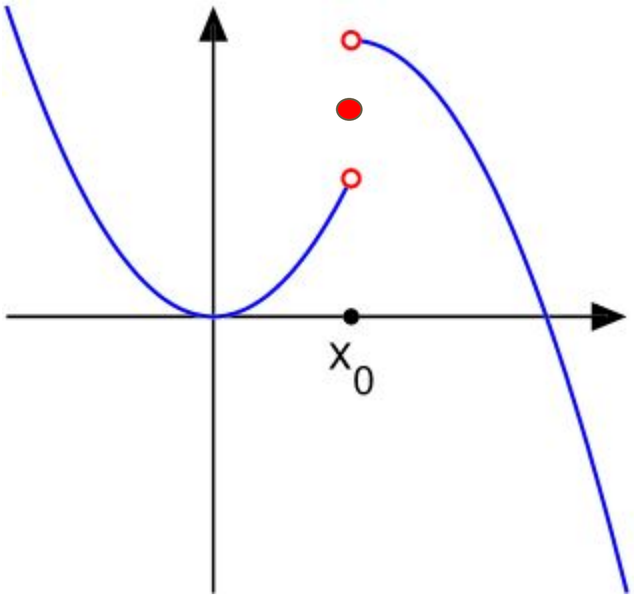
Why measure theory in probability?

The set of points where the Fourier series does not converge, or the set of events that violates the LLN, is an "exceptional yet negligible set."



Why measure theory in probability?

The event we only flip heads, or any event that violates the LLN,
has "measure zero," or "probability zero."



Why measure theory in probability?

$$\frac{X_1 + \ldots + X_n}{n} \longrightarrow \mathbb{E}X_1$$

If a property holds except on a set of measure zero, we say the property holds **"almost everywhere,"** or **"almost surely."**

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If a property holds except on a set of measure zero, we say the property holds **"almost everywhere,"** or **"almost surely."**

So the LLN holds with probability 1, or almost surely.

And we are allowed to say this thanks to measure theory.

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Besides the ruling out of problematic sets that we have explored, measure and integration are fundamental to the description of basic notions of probability such as expectation, random variables, convergence results, etc.

A Weird Problem

Problem 1

Suppose

$$X_n = \begin{cases} n^2 - 1, & \text{with probability } n^{-2}, \\ -1, & \text{with probability } 1 - n^{-2}. \end{cases}$$

Show that

$$\mathbb{E}[X_n] = 0 \quad \text{but} \quad n^{-1} S_n \to -1 \ \text{ a.s.}$$

Result from Measure Theory

Lemma 1

If

$$Z_k \geq 0 \quad \text{and} \quad \sum \mathbb{E}[Z_k] < \infty$$

Then

$$\sum Z_k < \infty \quad \text{(a.s)} \quad \text{so} \quad Z_k \to 0 \quad \text{(a.s)}$$

Borel Cantelli Lemma 1

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If Events E_n have

$$\sum \mathbb{P}[E_n] < \infty = 0$$

Then

$$\mathbb{P}[E_n \ \text{i.o}]$$

In other words,

$$\mathbb{P}[\{\omega : \omega \in E_n \ \text{for infinitely many n}\}] = 0$$

Back to Problem!

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Why does Strong Law of Large Numbers fail?

- What are the conditions for SLLN to hold?

Talk to the person next to you!

Why do we care about the Law of Large Numbers?

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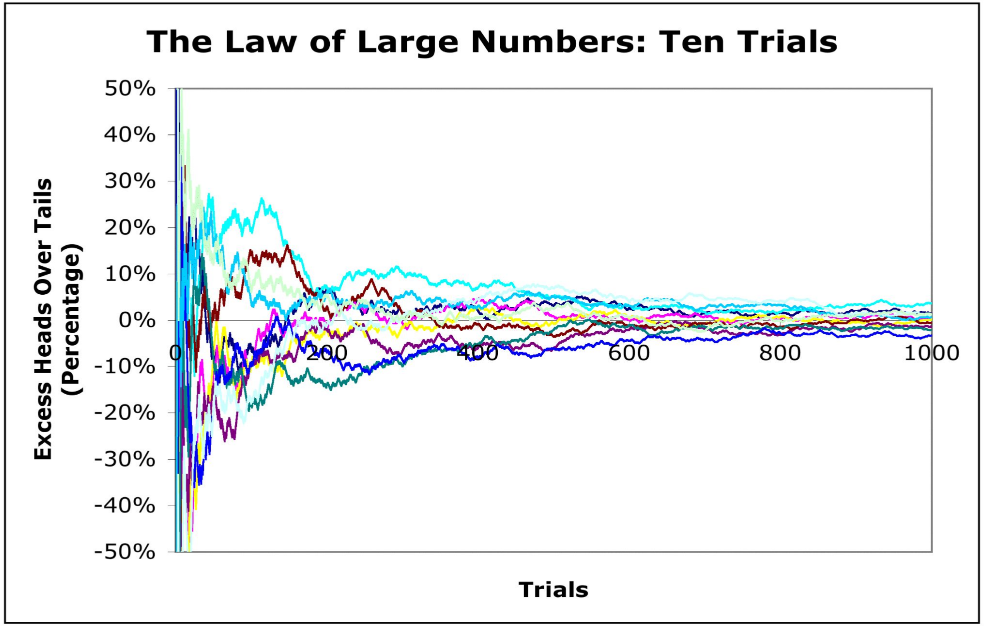

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Intuitively, adding randomness to an already random system should increase randomness—

But the opposite happens: randomness concentrates around typical behavior, making outcomes almost completely predictable.

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“A random variable … that depends on the influence of many independent variables is essentially constant.”

-Michel Talagrand

Concentration Principle

If $X_1, \ldots, X_n$ are independent (or weakly dependent) random variables, then the random variable $f(X_1, \ldots, X_n)$ is "close" to its mean $\mathbb{E}[f(X_1, \ldots, X_n)]$ provided that the function $f(x_1, \ldots, x_n)$ is not too "sensitive" to any of the coordinates x_i.

Source: Direct quote from the lecture notes (Van Handel, 2016)

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Concentration Phenomenon, intuitively.

This begs the question: What do we mean by “small”? How “small”?

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For simplicity, consider the case when $f(x_1, \ldots, x_n) = x_1 + \ldots + x_n$

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$$\mathbb{P}\Big(|S_n - \mathbb{E}S_n| \geq t\Big) = small$$

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But can we do smaller?

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Side note:

$$|F_n(x) - \Phi(x)| \leq \frac{C\rho}{\sigma^3\sqrt{n}}$$

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- etc.

A Concentration Result

Again, why should we care?

A Concentration Result

We illustrate with an example:

A Concentration Result

Consider the function $f : \mathbb{R}^n \to \mathbb{R}$ that maps $x \mapsto \|x\|_2$.

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A Concentration Result

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So the length of random vector X should be

$$\|X\|_2 \approx \sqrt{n}$$

(Recall that $Observed\ Value \approx Expected\ Value$)

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$$\mathbb{P}\Big(\big| \, \|X\|_2 - \sqrt{n} \big| \geq t \Big) = small$$

A Concentration Result

With a simple application of Bernstein's inequality we can get the following bound:

$$\mathbb{P}\Big(\big|\, \|X\|_2 - \sqrt{n}\big| \geq t\Big) \leq 2 \cdot exp(\frac{-ct^2}{2})$$

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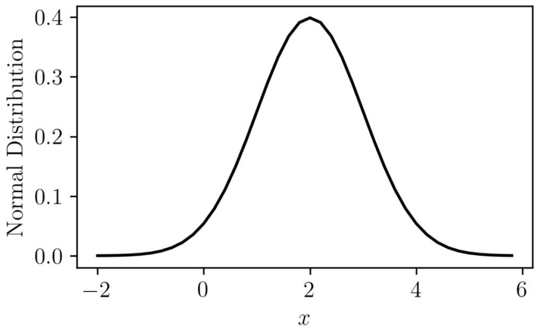
- The bound on our deviation does NOT depend on n. As n grows, almost all of our observations stay within a constant distance from $\sqrt{n}S^{n-1}$

A Concentration Result



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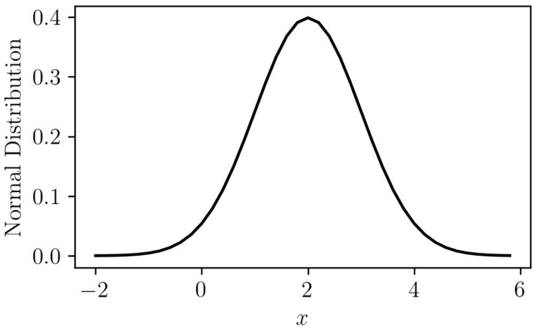

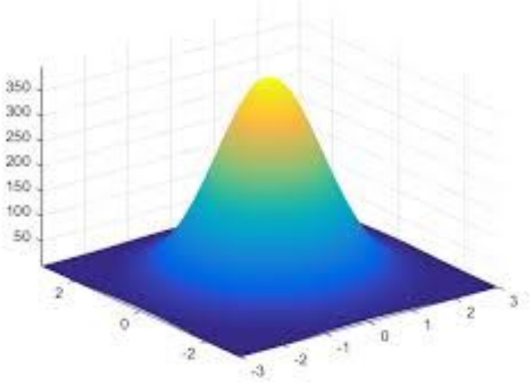

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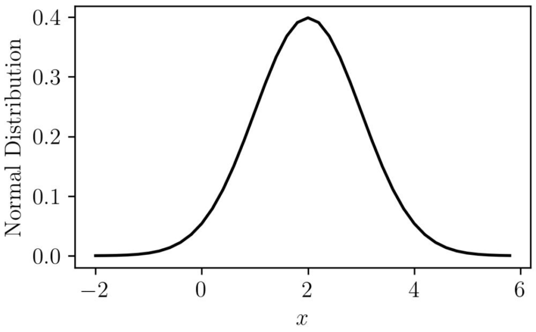

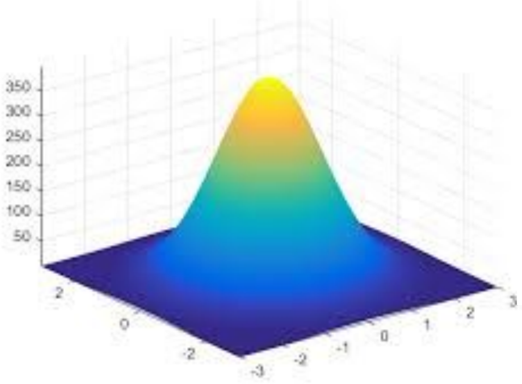

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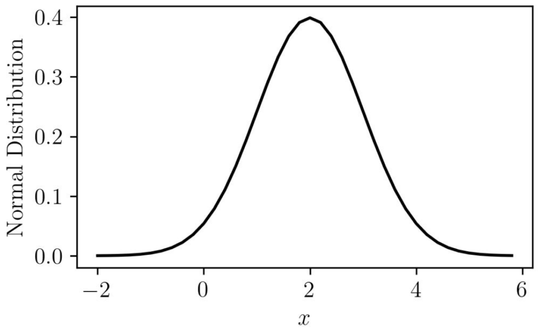

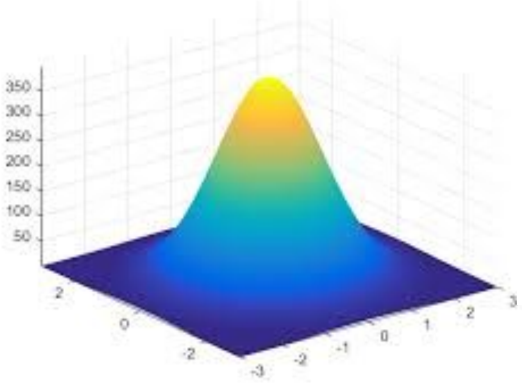

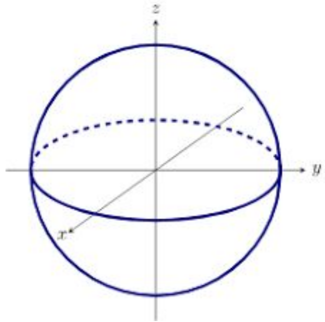

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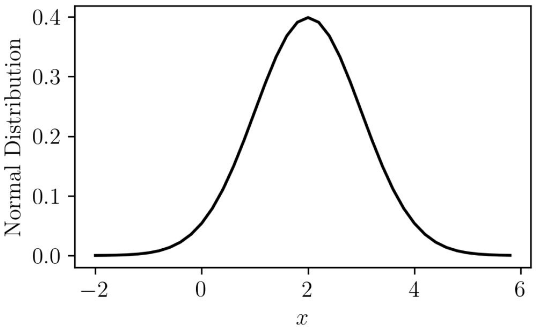
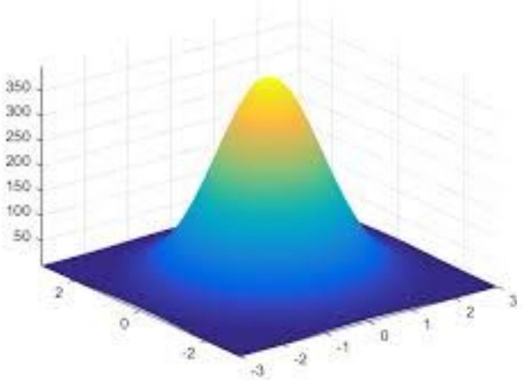
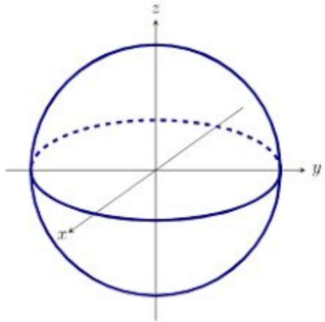
???







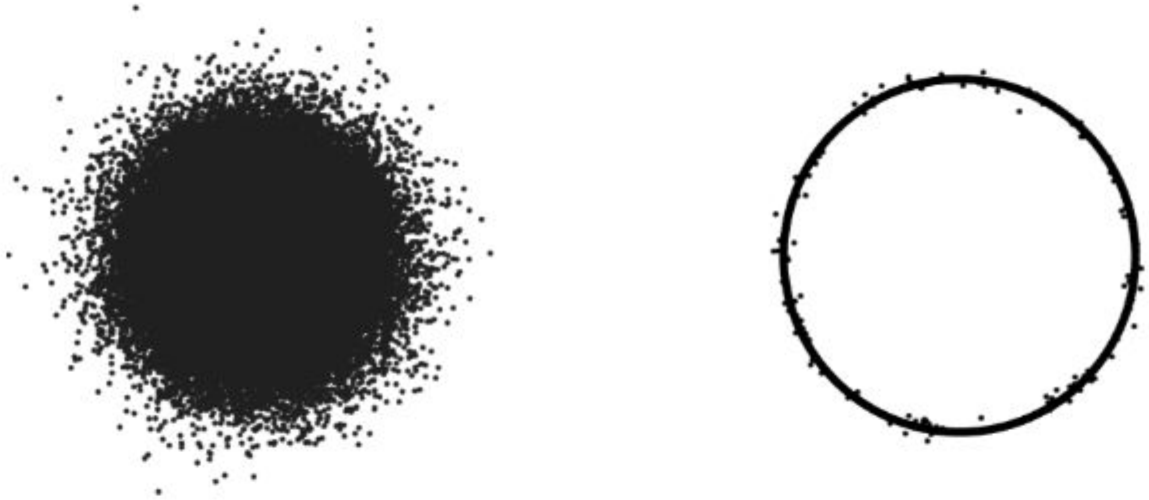


Figure 3.6 A Gaussian point cloud in two dimensions (left) and its intuitive visualization in high dimensions (right). In high dimensions, the standard normal distribution is very close to the uniform distribution on the sphere of radius $\sqrt{n}$.

Source: Vershynin, 2018

Applications to Statistics

Applications to Statistics

$$Observed\ Value \approx Expected\ Value$$

Applications to Statistics

$$Observed\ Value \approx Expected\ Value$$

$$Sample \approx Population$$ (With very high probability)

Applications to Statistics

- Covariance matrix estimation
- High-dimensional regression (e.g. LASSO)
- Empirical risk minimization
- etc.

Fun Fact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Acknowledgements

First and foremost, we want to thank Leon and Nila for being excellent mentors and for their time and effort in teaching us probability. We also want to thank Ethan for organizing the Directed Reading Program.

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Questions...?